

Newtonian vs. relativistic chaotic scattering

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 (Dated: April 26, 2010)

It is shown that Newtonian mechanics is not appropriate to compute approximate individual trajectories with small velocities in chaotic scattering. However, some global properties of the dynamical system, such as the dimension of the non-attracting chaotic invariant set, are more robust and the Newtonian approximation provides reasonably accurate results for them in slow chaotic scattering.

PACS numbers: 03.30.+p, 05.45.Ac, 05.45.-a

I. INTRODUCTION

It has been recently pointed out that, when the evolution is chaotic, the trajectories predicted by Newtonian mechanics are not good approximations to the special-relativistic trajectories, even for very small velocities, [1–3]. The problem considered in Refs. [1–3] is the evolution of periodically delta-kicked systems with or without damping. The purpose of this letter is twofold: we want to extend the previous conclusion to another area of deterministic chaos (chaotic scattering) but also to point out that, for slow motion, Newtonian mechanics can provide good approximations for global characteristics of chaotic systems, such as the structure and dimension of the non-attracting chaotic invariant set.

We will consider a relativistic particle moving in an external potential energy $V(\mathbf{r})$. If particle's mass, position and linear momentum are m , \mathbf{r} and \mathbf{p} , respectively, the Hamiltonian (i.e., the total energy) is $H = \sqrt{m^2 c^4 + c^2 \mathbf{p}^2} + V(\mathbf{r})$ and Hamilton's canonical equations read as follows:

$$\dot{\mathbf{r}} = \frac{\mathbf{p}}{m\gamma}, \quad \dot{\mathbf{p}} = -\nabla V(\mathbf{r}), \quad (1)$$

where the Lorentz factor is given by

$$\gamma \equiv \sqrt{1 + \frac{\mathbf{p}^2}{m^2 c^2}} = \left(1 - \frac{\dot{\mathbf{r}}^2}{c^2}\right)^{-1/2}. \quad (2)$$

The Newtonian equations of motion are recovered by putting $\gamma = 1$, so that one might think they are a good approximation for slow motion, where $\gamma \approx 1$. We will discuss in the next section to what extent is right this assumption in the case of chaotic scattering.

II. CHAOTIC SCATTERING

Let us consider a particle moving in the plane (x, y) under the force derived from the potential [4, 5]

$$V(x, y) = x^2 y^2 e^{-(x^2 + y^2)}. \quad (3)$$

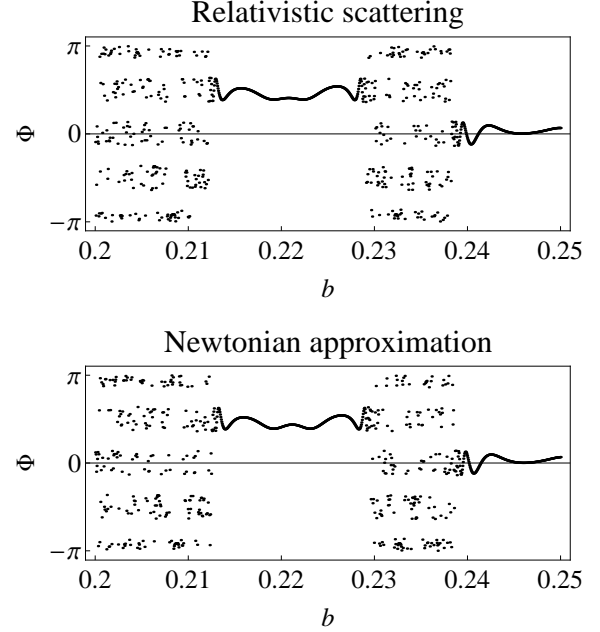


FIG. 1: Scattering angle $\Phi(b)$ for the relativistic and Newtonian equations of motion.

The particle is sent from $(x, y) = (-\infty, b)$ with initial velocity $(\dot{x}, \dot{y}) = (0, v)$ and after some time emerges from the scattering zone moving freely along a direction forming an angle Φ with the positive x axis. The initial velocity v is computed from the conserved energy $E = H - mc^2$. We choose a system of units in which $m = c = 1$ and a value of the energy $E = 0.05V_0$ —where $V_0 = e^{-2}$ is the height of the four maxima of the potential located at $(x, y) = (\pm 1, \pm 1)$ —, so that the motion is slow, since the Lorentz factor γ is always smaller than 1.007.

In Fig. 1 we have depicted 1000 values of the scattering function $\Phi(b)$, in the range $[0.2, 0.25]$ of the impact parameter b , obtained by numerical integration [6] of the relativistic equations of motion (1) and their Newtonian approximation.

First of all, one can see that while Newtonian mechanics gives reasonably approximate results when the scattering function is smooth, when the latter is discontinuous the predictions of relativistic and Newtonian equations of motions are very often completely different. This extends to chaotic

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scattering the claim in Refs. [1–3] that Newtonian equation of motions are not a good approximation when *individual* chaotic trajectories have to be computed.

However, although the positions of individual points in the regions of chaotic scattering may be very different, the position of the discontinuity points along the b axis seems to be very similar in both cases. This statement can be made quantitative if one computes the dimension of the Cantor set of discontinuity points by using the method of the uncertainty exponent [4, 5]. We have computed the scattering function for 2^{22} values of $b \in [0.2, 0.25]$ and then selected a small value ϵ . We say that a value b is *certain* if for the three impact parameters $b - \epsilon$, b and $b + \epsilon$ the scattering is upward or if it is downward for the three values. We repeat this for many values of b and call $f(\epsilon)$ to the proportion of uncertain values of b . Figure 2 show a log-log plot of several values of $f(\epsilon)$ along with a good linear fit, so that $f(\epsilon) \sim e^\alpha$ with a uncertainty exponent given by the slope of the straight line: $\alpha \approx 0.139$. In consequence [4, 5], the fractal dimension of the set of discontinuity points is $D_0 = 1 - \alpha \approx 0.861$. The corresponding plot for the Newtonian approximation looks nearly identical with $\alpha \approx 0.141$. This shows that the dimension of the chaotic invariant set is well approximated when computed with Newtonian mechanics, if the velocities are small enough.

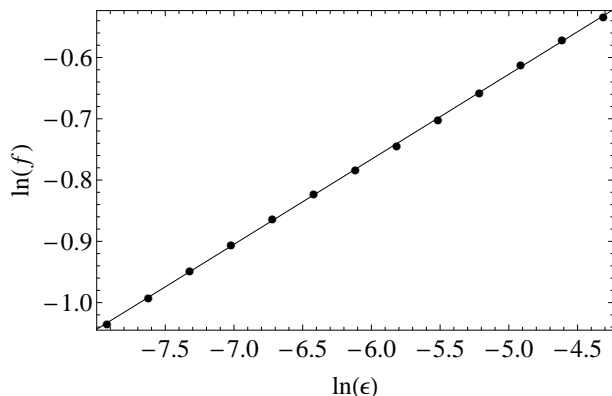


FIG. 2: Log-log plot of some points of the uncertainty function $f(\epsilon)$ and a linear fit.

III. FINAL COMMENTS

In the previous section we have used Newtonian mechanics in a consistent way, so that for each impact parameter b the initial velocity v is computed using the Newtonian definition for the kinetic energy. Since the latter is different in special relativity, slightly different initial velocities were used in the Newtonian and the relativistic simulations. If one uses the same initial velocities in both cases (which implies slightly different energies), the results are very similar to those discussed above. For instance, one gets $\alpha \approx 0.142$.

We have shown that also in chaotic scattering Newtonian mechanics fails to be a good approximation to compute individual trajectories. Even for slow motion, the little relativistic corrections to the equations of motion, which remain small in non-chaotic scattering, are greatly amplified by chaotic scattering during the long time in which the trajectory remains close to the non-attracting invariant set. This extends to chaotic scattering the conclusions reached in Refs. [1–3]. However, we have also shown that some global properties, such as the structure and dimension of the set of discontinuity points (and, thus, the dimension of the non-attracting chaotic invariant set) are more robust and Newtonian mechanics provides reasonably accurate approximations to them if the motion is slow enough.

Acknowledgments

This work was supported by the Basque Government (Department of Education, Universities and Research) (Research Grant GIC07/51-IT-221-07).

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